Evolution equation in infinite dimensional distribution spaces

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Hochberg has considered the parabolic partial differential equation

$$\frac{\partial u}{\partial t} = (-1)^{n+1} \frac{\partial^{2n} u}{\partial x^{2n}}, \quad n \geq 2.$$

He prove that the fundamental solution of this equation is a additive signed measure.

In this Talk we will consider **the infinite dimensional generalization of the above equation** with an additional potential function V_t

$$\frac{\partial U_t}{\partial t} = (-1)^{p+1} \frac{1}{2} (\Delta_G)^p U_t + V_t, \quad p \in \mathbb{N}, \tag{1}$$

where Δ_G is the Gross Laplacian . We will study this equation with an initial condition and show that the unique solution is a generalized function.The main tool is the interpretation of the Gross Laplacian as a convolution operator.

Preliminary

Let X be a real nuclear Fréchet space with topology given by an increasing family $\{|\cdot|_p;\, p \in \mathbb{N}_0\}$ of Hilbertian norms, $\mathbb{N}_0$ being the set of nonnegative integers. Then X is represented as

$$X = \bigcap_{p \in \mathbb{N}_0} X_p,$$

where the Hilbert space X_p is the completion of X with respect to the norm $|\cdot|_p$. We use X_{-p} to denote the dual space of X_p. Then the dual space X' of X can be represented as $X' = \bigcup_{p \in \mathbb{N}_0} X_{-p}$ and is equipped with the inductive limit topology.

Let

$$N = X + iX$$

and

$$N_p = X_p + iX_p,\ p \in \mathbb{Z},$$

be the complexifications of X and X_p, respectively.
For $n \in \mathbb{N}_0$ we denote by $N^{\widehat{\otimes} n}$ the n-fold symmetric tensor product of N equipped with the π-topology and by $N_p^{\widehat{\otimes} n}$ the n-fold symmetric Hilbertian tensor product of N_p. We will preserve the notation $|\cdot|_p$ and $|\cdot|_{-p}$ for the norms on $N_p^{\widehat{\otimes} n}$ and $N_{-p}^{\widehat{\otimes} n}$, respectively.

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Young functions

Let θ be a **Young function**, i.e., it is a continuous, **convex,** and increasing function defined on $\mathbb{R}_+$ such that $\theta(0) = 0$ and

$$\lim_{x \to \infty} \theta(x)/x = \infty$$

We define **the conjugate function** (or the Legendre Transform) θ^* of θ by

$$\theta^*(x) = \sup_{t \geq 0} \left(tx - \theta(t)\right), \quad x \geq 0.$$

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Functional Spaces

For a Young function θ, we denote by $\mathcal{F}_\theta(N')$ the space of **holomorphic functions** on N' with **exponential growth** of order θ and of minimal type.
Similarly, let $\mathcal{G}_\theta(N)$ denote the space of holomorphic functions on N with exponential growth of order θ and of arbitrary type.
Moreover, for each $p \in \mathbb{Z}$ and $m > 0$, define

$$\mathrm{Exp}(N_p, \theta, m)$$

to be the space of entire functions f on N_p satisfying the condition:

$$\|f\|_{\theta,p,m} = \sup_{x \in N_p} |f(x)| e^{-\theta(m|x|_p)} < \infty.$$

Functional Spaces

Then the spaces $\mathcal{F}_\theta(N')$ and $\mathcal{G}_\theta(N)$ can be represented as

$$\mathcal{F}_\theta(N') = \bigcap_{p\in\mathbb{N}_0,\, m>0} \mathrm{Exp}(N_{-p}, \theta, m),$$

$$\mathcal{G}_\theta(N) = \bigcup_{p\in\mathbb{N}_0,\, m>0} \mathrm{Exp}(N_{p}, \theta, m),$$

and are equipped with the projective limit topology and the inductive limit topology, respectively.
The space $\mathcal{F}_\theta(N')$ is called the space of *test functions* on N'. Its dual space $\mathcal{F}'_\theta(N')$, equipped with the strong topology, is called the space of *distributions* on N'.

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Formal Power Series Spaces

For $p \in \mathbb{N}_0$ and $m > 0$, we define the Hilbert spaces

$$F_{\theta,m}(N_p) = \left\{ \vec{\varphi} = (\varphi_n)_{n=0}^{\infty} \; ; \; \varphi_n \in N_p^{\widehat{\otimes} n}, \sum_{n\geq 0} \theta_n^{-2} m^{-n} |\varphi_n|_p^2 < \infty \right\},$$

$$G_{\theta,m}(N_{-p}) = \left\{ \vec{\Phi} = (\Phi_n)_{n=0}^{\infty} \; ; \; \Phi_n \in N_{-p}^{\widehat{\otimes} n}, \sum_{n\geq 0} (n!\theta_n)^2 m^n |\Phi_n|_{-p}^2 < \infty \right\},$$

where

$$\theta_n = \inf_{r>0} e^{\theta(r)}/r^n, \; n \in \mathbb{N}_0$$

Put

$$F_\theta(N) = \bigcap_{p\in\mathbb{N}_0, m>0} F_{\theta,m}(N_p),$$

$$G_\theta(N') = \bigcup_{p\in\mathbb{N}_0, m>0} G_{\theta,m}(N_{-p}).$$

Theorem

The space $F_\theta(N)$ equipped with the projective limit topology is a nuclear Fréchet space.

The space $G_\theta(N')$ carries the dual topology of $F_\theta(N)$ with respect to the $\mathbb{C}$-bilinear pairing given by

$$\langle\langle \vec{\Phi}, \vec{\varphi} \rangle\rangle = \sum_{n \geq 0} n! \langle \Phi_n, \varphi_n \rangle, \quad (2)$$

where

$$\vec{\Phi} = (\Phi_n)_{n=0}^{\infty} \in G_\theta(N')$$

and

$$\vec{\varphi} = (\varphi_n)_{n=0}^{\infty} \in F_\theta(N)$$

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Taylor Map

It was proved by Gannoun-Hachaichi-Rezgui-Ouerdiane that the **Taylor map** defined by

$$T: \ \varphi \longmapsto \left(\frac{1}{n!}\varphi^{(n)}(0)\right)_{n=0}^{\infty}$$

is a topological isomorphism :

$$\mathcal{F}_\theta(N') \mapsto F_\theta(N)$$

Theorem

The Taylor map T establish a topological isomorphism from $\mathcal{G}_{\theta^*}(N))$ onto $G_\theta(N'))$.

The action of a distribution $\Phi \in \mathcal{F}'_\theta(N')$ on a test function $\varphi \in \mathcal{F}_\theta(N')$ can be expressed in terms of the Taylor map as follows:

$$\langle\langle \Phi, \varphi \rangle\rangle = \langle\langle \vec{\Phi}, \vec{\varphi} \rangle\rangle,$$

where

$$\vec{\Phi} = (T^*)^{-1}\Phi$$

and

$$\vec{\varphi} = T\varphi$$

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Laplace Transform

It is easy to see that for each $\xi \in N$, **the exponential function**

$$e_\xi(z) = e^{\langle z,\xi\rangle}, \quad z \in N',$$

is a test function in the space $\mathcal{F}_\theta(N')$ for any Young function θ. Thus we can define the ***Laplace transform*** of a distribution $\Phi \in \mathcal{F}'_\theta(N')$ by

$$\widehat{\Phi}(\xi) = \langle\langle \Phi, e_\xi \rangle\rangle, \quad \xi \in N. \tag{3}$$

Theorem

The Laplace transform is a topological isomorphism from $\mathcal{F}'_\theta(N')$ onto $\mathcal{G}_{\theta^*}(N)$.

Convolution Calculus

For $\varphi \in \mathcal{F}_\theta(N')$, the *translation* $\mathrm{t}_x\varphi$ of φ by $x \in N'$ is defined by

$$\mathrm{t}_x\varphi(y) = \varphi(y-x), \quad y \in N'.$$

The translation operator t_x is a continuous linear operator from $\mathcal{F}_\theta(N')$ into itself for any $x \in N'$.
By a ***convolution operator*** on the space $\mathcal{F}_\theta(N')$ of test functions we mean a continuous linear operator from $\mathcal{F}_\theta(N')$ into itself which commutes with translation operators t_x for all $x \in N'$.

Definition

We define the *convolution*

$$\Phi * \varphi$$

of a distribution $\Phi \in \mathcal{F}'_\theta(N')$ and a test function $\varphi \in \mathcal{F}_\theta(N')$ to be the function

$$(\Phi * \varphi)(x) = \langle\langle \Phi, \mathrm{t}_{-x}\varphi \rangle\rangle, \quad x \in N'.$$

Lemma

Let $\Phi \in \mathcal{F}'_\theta(N')$ arbitrarily fixed. Then for any $\varphi \in \mathcal{F}_\theta(N')$, $\Phi * \varphi \in \mathcal{F}_\theta(N')$ and the mapping T_Φ defined by

$$T_\Phi : \varphi \longmapsto \Phi * \varphi, \quad \varphi \in \mathcal{F}_\theta(N'),$$

is a convolution linear operator on $\mathcal{F}_\theta(N')$.

Convolution operators

Conversely, it was proved by Ben Chrouda-Oued-Ouerdiane that all convolution operators on $\mathcal{F}_\theta(N')$ occur this way, i.e., if T is a convolution operator on $\mathcal{F}_\theta(N')$, then there exists a unique $\Phi \in \mathcal{F}'_\theta(N')$ such that

$$T = T_\Phi$$

or equivalently,

$$T(\varphi) = T_\Phi(\varphi) = \Phi * \varphi, \quad \varphi \in \mathcal{F}_\theta(N'). \tag{4}$$

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Convolution of two Distributions

Suppose $\Phi_1, \Phi_2 \in \mathcal{F}'_\theta(N')$. Let T_{Φ_1} and T_{Φ_2} be the convolution operators given by Φ_1 and Φ_2, respectively, as in Equation (4). It is clear that the composition

$$T_{\Phi_1} \circ T_{\Phi_2}$$

is also a convolution operator on $\mathcal{F}_\theta(N')$. Hence there exists a unique distribution, denoted by

$$\Phi_1 * \Phi_2$$

in $\mathcal{F}'_\theta(N')$ such that

$$T_{\Phi_1} \circ T_{\Phi_2} = T_{\Phi_1 * \Phi_2}. \tag{5}$$

Theorem

For any $\Phi_1, \Phi_2 \in \mathcal{F}'_\theta(N')$, the distribution $\Phi_1 * \Phi_2$ in Equation (5) is called the convolution of Φ_1 and Φ_2, and we have the following equality via the Laplace transform:

$$(\Phi_1 * \Phi_2)\widehat{\ } = \widehat{\Phi}_1\, \widehat{\Phi}_2. \tag{6}$$

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White Noise Gel'fand Triple

Let γ be the standard Gaussian measure on the dual space X' of the real nuclear space X, namely, its characteristic function is given by

$$\int_{X'} e^{i\langle y,\xi\rangle}\, d\gamma(y) = e^{-|\xi|_0^2/2}, \quad \xi \in X,$$

where $|\cdot|_0$ is the norm $|\cdot|_p$ on X for $p = 0$.

Theorem

Suppose that the Young function θ satisfies the condition:

$$\lim_{r\to+\infty} \frac{\theta(r)}{r^2} < +\infty$$

Then we obtain the Gel'fand triple

$$\mathcal{F}_\theta(N') \hookrightarrow L^2(X',\gamma) \hookrightarrow \mathcal{F}'_\theta(N').$$

Remark

In the White Noise theory we consider the particular case $X = S(\mathbb{R})$, and $\theta(x) = x^2$ and use usually the S-Transform to characterize Hida distributions in term of analytic functionals with growth conditions (see for example the works of T. Hida, Y. Kondratiev, Y. Potthoff, L. Streit, H. H; Kuo, N. Obata and H Ouerdiane (2011),B. Oksendal, Y. J. Lee, U.C . Ji ...).

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Let β be a continuous, convex, and increasing function on $\mathbb{R}^+$. Suppose f is function in $\mathrm{Exp}(\mathbb{C}, \beta, m)$ for some $m > 0$. For each distribution Φ in $\mathcal{F}'_\theta(N')$, we define the *convolution composition* $f^*(\Phi)$ of f and Φ by

$$\left(f^*(\Phi)\right)^\frown = f(\widehat{\Phi}). \quad (7)$$

Its easy to see that $f^*(\Phi)$ belongs to $\mathcal{F}'_\lambda(N')$ with

$$\lambda = (\beta \circ e^{\theta^*})^*$$

In particular, when

$$\beta(x) = x,\ x \in \mathbb{R}_+,$$

and

$$f(z) = e^z,\ z \in \mathbb{C}$$

we get a distribution $e^{*\Phi}$ in the space $\mathcal{F}'_{(e^{\theta^*})^*}(N')$ for each $\Phi \in \mathcal{F}'_\theta(N')$. Moreover, by Equation (7), we have

$$(e^{*\Phi})^{\widehat{}} = e^{\widehat{\Phi}}. \qquad (8)$$

Theorem

The distribution $e^{*\Phi}$ has the following series expansion

$$e^{*\Phi} = \sum_{n=0}^{\infty} \frac{1}{n!} \Phi^{*n},$$

where $\Phi^{*n} = \Phi * \Phi * \cdots * \Phi$ (n times) and the convergence is in $\mathcal{F}'_{(e^{\theta^*})^*}(N')$ with respect to the strong topology.

Initial-valued Evolution equation

Let $I \subset \mathbb{R}$ be an interval containing the origin. Consider a family

$$\{\Phi_t\,;\ t \in I\}$$

of distributions in $\mathcal{F}'_\theta(N')$. We assume that the function

$$t \mapsto \Phi_t$$

is continuous from I into $\mathcal{F}'_\theta(N')$. Then the function

$$t \mapsto \widehat{\Phi_t}$$

is continuous from I into $\mathcal{G}_{\theta^*}(N)$. Thus for each $t \in I$, the set

$$\{\widehat{\Phi_s}\,;\ s \in [0, t]\}$$

is a compact subset of $\mathcal{G}_{\theta^*}(N)$. In particular, it is bounded in $\mathcal{G}_{\theta^*}(N)$.

Hence there exist constants $p \in \mathbb{N}_0,\ m > 0$, and $C_t > 0$ such that

$$|\widehat{\Phi_s}(\xi)| \leq C_t\, e^{\theta^*(m|\xi|_p)}, \quad \forall\, s \in [0,t] \text{ and } \xi \in N_p.$$

This inequality shows that the function

$$\xi \mapsto \int_0^t \widehat{\Phi_s}(\xi)\, ds$$

belongs to the space $\mathcal{G}_{\theta^*}(N)$.

Lemma

There exists a unique distribution, denoted by $\int_0^t \Phi_s\, ds$, in $\mathcal{F}'_\theta(N')$ satisfying

$$\left(\int_0^t \Phi_s\, ds\right)^{\widehat{}}(\xi) = \int_0^t \widehat{\Phi_s}(\xi)\, ds, \quad \xi \in N.$$

Moreover, the process

$$E_t = \int_0^t \Phi_s\, ds,\ t \in I,$$

is differentiable in $\mathcal{F}'_\theta(N')$ and satisfies the equation

$$\frac{\partial}{\partial t} E_t = \Phi_t.$$

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Main theorem

Let $\{\Phi_t\}$ and $\{M_t\}$ be two continuous $\mathcal{F}'_\theta(N')$-processes. Consider the initial value problem

$$\frac{dX_t}{dt} = \Phi_t * X_t + M_t\,, \quad X_0 = F \in \mathcal{F}'_\theta(N'). \tag{9}$$

Theorem

The stochastic differential equation (9) *has a unique solution in* $\mathcal{F}'_{(e^{\theta*})^*}(N')$ *given by*

$$X_t = F * e^{*\int_0^t \Phi_s\, ds} + \int_0^t e^{*\int_s^t \Phi_u\, du} * M_s\, ds. \tag{10}$$

We can apply the previous Theorem to study an evolution equation for a power of the Gross Laplacian and a generalized potential function with the initial condition being a generalized function

Gross Laplacian

Let $\varphi \in \mathcal{F}_\theta(N')$ be represented by

$$\varphi(x) = \sum_{n\geq 0} \langle x^{\otimes n}, \varphi^{(n)} \rangle.$$

The Gross Laplacian $(\Delta_G \varphi)(x)$ of φ at $x \in N'$ is defined to be

$$(\Delta_G \varphi)(x) = \sum_{n\geq 0} (n+2)(n+1) \langle x^{\otimes n}, \langle \tau, \varphi^{(n+2)} \rangle \rangle,$$

where τ is the trace operator, namely,

$$\langle \tau, \xi \otimes \eta \rangle = \langle \xi, \eta \rangle, \quad \xi, \eta \in N.$$

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Gross Laplacian as a convolution operator

It turns out that the Gross Laplacian Δ_G can be extended to be a continuous linear operator from $\mathcal{F}'_\theta(N')$ into itself and his extension to generalized functions is a convolution operator :

Theorem

For any $\Psi \in \mathcal{F}'_\theta(N')$,

$$\Delta_G \Psi = \mathcal{T} * \Psi, \tag{11}$$

where $\mathcal{T}$ is the generalized function in $\mathcal{F}'_\theta(N')$ with the Formal power series is given by

$$\vec{\mathcal{T}} = (0, 0, \tau, 0, \cdots) \in G_\theta(N')$$

as in Equation (2).

Theorem

For every positive integer p we have

$$\Delta_G^p \Psi = (\mathcal{T}^{*p}) * \Psi, \quad \Psi \in \mathcal{F}'_\theta(N'). \tag{12}$$

Moreover, the generalized function associated with Δ_G^p is given by

$$\overrightarrow{\mathcal{T}^{*p}} = (0, 0, \ldots, \tau^{\otimes p}, 0, \ldots). \tag{13}$$

Proof

Proof. Using Equations (5) and (11), we obtain

$$\Delta_G^p \Psi = \mathcal{T}^{*p} * \Psi$$

But the Laplace transform of $\mathcal{T}$ is given by

$$\widehat{\mathcal{T}}(\xi) = \langle \tau, \xi^{\otimes 2} \rangle = \langle \xi, \xi \rangle = |\xi|_0^2.$$

Hence we have

$$\widehat{(\mathcal{T}^{*p})}(\xi) = \langle \tau, \xi^{\otimes 2} \rangle^p = \langle \xi, \xi \rangle^p = |\xi|_0^{2p}.$$

For any positive integer p, let $S = \mathcal{T}^{*p}$ and let the formal power series associated with S be given by $\widehat{S} = (S_0, S_1, \ldots, S_n, \ldots)$. Then we can use the definition of the Laplace transform and the bilinear pairing between test functions and distributions in Equation (2) to deduce the following relationship

$$\widehat{(\mathcal{T}^{*p})}(\xi) = \langle \mathcal{T}^{*p}, e^{\xi} \rangle = \sum_{n \geq 0} n! \langle S_n, \frac{\xi^{\otimes n}}{n!} \rangle = \langle \xi, \xi \rangle^p,$$

which implies that $S_n = 0$ for all $n \neq 2p$ and $S_{2p} = \tau^{\otimes p}$.
Therefore,

$$\overrightarrow{S} = \overrightarrow{\mathcal{T}^{*p}} = (0, 0, \ldots, \tau^{\otimes p}, 0, \ldots).$$

This proves Equation (13). □

Theorem

Let θ be a Young function such that $\lim_{r\to\infty}\theta(r)/r^2 < \infty$ and $F \in \mathcal{F}'_\theta(N')$. Then the following evolution equation associated with the p-th power of the Gross Laplacian and a continuous $\mathcal{F}'_\theta(N')$-valued potential function V_t

$$\frac{\partial U_t}{\partial t} = (-1)^{p+1}\frac{1}{2}\Delta_G^p U_t + V_t, \quad U_0 = F, \tag{14}$$

has a unique solution in the space $\mathcal{F}'_\theta(N')$ given by

$$U_t = F * e^{*\frac{t}{2}(-1)^{p+1}\mathcal{T}^{*p}} + \int_0^t e^{*\frac{t-s}{2}(-1)^{p+1}\mathcal{T}^{*p}} * V_s\, ds, \tag{15}$$

where $\mathcal{T}$ is the generalized function given by Equation (11).

Proof

Proof. Use Equation (11) to rewrite Equation (14) as

$$\frac{\partial U_t}{\partial t} = (-1)^{p+1}\frac{1}{2}\mathcal{T}^{*p} * U_t + V_t, \quad U_0 = F.$$

Then we can apply Theorem 10 to this equation to get the unique solution in Equation (15).

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Interpretation of the solutions of the evolution equation

For $t > 0$, define the distribution $\mu_{t,p}$ defined by its Laplace transform

$$\widehat{\mu_{t,p}}(\xi) = \exp\left[\frac{(-1)^{p+1}t}{2}\langle \xi, \xi \rangle^p\right], \quad \xi \in N. \tag{16}$$

From the duality theorem which states that the Laplace transform is a topological isomorphism from $\mathcal{F}'_\theta(N')$ onto $\mathcal{G}_{\theta^*}(N)$. Hence Equation (16) implies that $\mu_{t,p}$, $t > 0$, are generalized functions in the space $\mathcal{F}'_\theta(N')$ with the Young function given by

$$\theta(x) = x^{\frac{2p}{2p-1}}, \quad x \geq 0.$$

Therefore, the solution U_t in equation (15) can be rewritten as

$$U_t = F * \mu_{t,p} + \int_0^t \mu_{t-s,p} * V_s \, ds.$$

In particular, when $V_t = 0$, we have the evolution equation

$$\frac{\partial U_t}{\partial t} = (-1)^{p+1} \frac{1}{2} \Delta_G^p U_t, \quad U_0 = F, \tag{17}$$

which has a unique solution given by

$$U_t = F * \mu_{t,p}. \tag{18}$$

Hochberg has studied the one-dimensional case of Equation (17) and showed that the fundamental solution defines a finitely additive measure with unbounded total variation. Using the white noise theory, we can now interpret this “finitely additive measure with unbounded total variation" as a generalized function in the space $\mathcal{F}'_\theta(N')$, which is given by Equation (18).

This phenomenon is very much like the case of Feynman integral, which had been regarded as a finitely additive measure with unbounded total variation before the theory of white noise was introduced by T. Hida in 1975. It is a well-known fact that the Feynman integral is a generalized function.

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When $p = 1$, Equation (16) gives the equality

$$\widehat{\mu_{t,1}}(\xi) = \exp\left[-\frac{t}{2}|\xi|_0^2 \right], \quad \xi \in X,$$

which shows that $\mu_{t,1}$ is the standard Gaussian measure on X' with variance t, i.e., $\mu_{t,1} = \gamma_t$ with γ_t defined by

$$\gamma_t(\cdot) = \gamma\Big(\frac{\cdot}{\sqrt{t}}\Big).$$

Note that the probability measure $\mu_{t,1}$ induces a positive distribution in the space $\mathcal{F}'_\theta(N')$ given by

$$\langle\langle \mu_{t,1} \varphi \rangle\rangle = \int_{X'} \varphi(x)\, d\mu_{t,1}(x) = \int_{X'} \varphi(\sqrt{t}\, x)\, d\gamma(x), \quad \varphi \in \mathcal{F}_\theta(N').$$

. Moreover, if the potential function is given by $V_t = \alpha \dot{W}_t$ with $\alpha \in \mathbb{R}$ and $\dot{W}_t$ a white noise, then the solution in Equation (15) reduces to the one obtained by Barhoumi-Kuo-Ouerdiane.

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